A DIFFERENCE PROPERTY FOR POLYNOMIALS AND EXPONENTIAL POLYNOMIALS ON ABELIAN LOCALLY COMPACT GROUPS [1]

BY

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1. **Introduction.** Let f be a complex-valued function on E^m, Euclidean m-space, which has the property that for each $h \in E^m$, the function $\Delta_h f$: $\Delta_h f(x) = f(x+h) - f(x)$ is continuous (or is a polynomial, or an exponential polynomial). Then f itself need not be continuous (or a polynomial, or an exponential polynomial), for there exist nonmeasurable *additive* functions on E^m, that is, nonmeasurable solutions Γ of the functional equation $\Gamma(x+y) = \Gamma(x) + \Gamma(y)$. However, de Bruijn [1], [2] (for E^1) and Kemperman [7], [8] (for E^m) showed that, among many others, the classes of continuous functions, polynomials, and certain classes of trigonometric and exponential polynomials have the property that if $\Delta_h f$ is in the class for each $h \in E^m$, then there exists an additive function Γ on E^m such that $f - \Gamma$ is in the class.

Let G be an abelian topological group, and let Ω be a class of complex-valued functions on G which contains the constant functions, and such that $f,g \in \Omega$ implies $f - g \in \Omega$ and $f_h \in \Omega$ for each $h \in G$, where $f_h(x) = f(x+h)$. The class Ω is said to have the *difference property* if the following implication holds: let f be a complex-valued function on G such that $\Delta_h f \in \Omega$ for each $h \in G$. Then there is an additive function Γ on G such that $f - \Gamma \in \Omega$.

Except where the contarary is explicitly stated, G will denote an abelian locally compact group. The product of m copies of the reals will be denoted by E^m, and the group of integers by C. All functions considered are complex-valued. It is known [3] that the class of continuous functions on G has the difference property. The principal results of this paper are Theorems 1 and 2, which give necessary and sufficient conditions on G in order that the class of polynomials on G (as defined, for instance, in [6]) and the class of exponential polynomials on G have the difference property.

2. **The difference property for polynomials.** *A* function P on G is a *polynomial of degree* N $(N < \infty)$, provided

(P1) for each $(a,b) \in G \times G$, the function

$$P_{ab}: P_{ab}(\lambda) = P(a + \lambda b) \qquad (\lambda \in C)$$

is a polynomial on C.

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(P2) if $N_{a,b}$ denotes the degree of $P_{a,b}$, then $N = \max\{N_{a,b}: (a,b) \in G \times G\}$, and

(P3) P is continuous on G.

An element $b \in G$ is said to be *compact* if the closure of the subgroup generated by b is compact. The set B of all compact elements is a closed subgroup [11]. Since $G = E^m + G'$, where G' contains a compact open subgroup [12], it follows that $G/B = E^m + G_1$, where G_1 is discrete. If a function P has properties (P1) and (P3) on G, if $a \in G$ and $b \in B$, then the set of polynomial values $\{P_{ab}(\lambda): \lambda \in C\}$ is bounded, since $\{a + \lambda b: \lambda \in C\}$ has compact closure. Thus $P_{ab}(\lambda) \equiv P(a)$, so that P is constant on cosets of B, and is essentially a function on G/B.

A set $H \subset G$ is a *Hamel basis* for G provided that for no finite subset $\{b_0, b_1, \cdots, b_k\}$ of H do there exist integers $N \neq 0$, $n_1, \cdots, n_k$ such that $Nb_0 = n_1 b_1 + \cdots + n_k b_k$, and provided that H is maximal with respect to this property. It follows from the Hausdorff maximal principle that every abelian group has a (possibly empty) Hamel basis. If $x \in G$, then there exist uniquely determined elements $b_1, \cdots, b_k$ in H and integers $N \neq 0$, $n_1, \cdots, n_k$ such that

$$Nx = n_1 b_1 + \cdots + n_k b_k. \tag{2.1}$$

If (2.1) holds, we shall write

$$x \sim (n_1/N) b_1 + \cdots + (n_k/N) b_k = r_1(x) b_1 + \cdots + r_k(x) b_k.$$

In this way there is associated with each $b_\alpha \in H$ a rational-valued function r_α on G, and r_α is easily seen to be additive.

THEOREM 1. *Let G be an abelian locally compact group, and let B be the group of compact elements of G. A necessary and sufficient condition in order that the class of polynomials on G have the difference property is that $G/B = E^m + G_1$, where G_1 has a finite Hamel basis.*

The following example shows the necessity of the condition.

EXAMPLE 1. Let $G/B = G_1 + E^m$ where G_1 is discrete and has an infinite Hamel basis. Then there exists a continuous function on G which is not a polynomial, but each of whose differences is a polynomial. It suffices to show that such a function f exists on G_1, for then the function

$$f_1: f_1(x) = f(P_2 P_1 x),$$

where P_1 and P_2 are the natural mappings from G to G/B and from G/B to G_1 respectively, is an extension of f which has the required properties on G. Let $b_1, b_2, \cdots$ be a countably infinite subset of the Hamel basis of G_1, let $r_1, r_2, \cdots$ be the corresponding rational-valued functions, and let

$$f(x) = \sum_{\nu=1}^{\infty} (r_\nu(x))^\nu \qquad (x \in G_1).$$

If h is an arbitrary fixed element of G_1, there is an integer $N=N(h)$ such that $\nu > N$ implies $r_\nu(h) = 0$. Thus, from the additivity of the r_ν,

$$f(x+h)-f(x)=\sum_{\nu=1}^{N}\left[(r_\nu(x)+r_\nu(h))^\nu-(r_\nu(x))^\nu\right],$$

a polynomial.

It may be noticed that the function f in Example 1 satisfies (P1). In fact, the following is true: *if G is an arbitrary abelian group, and if P is a function on G such that, for each $(a,b)\in G\times G$, the function given by $\Delta_b P(a+\lambda b)$ $(\lambda\in C)$ is a polynomial on C, then P has property* (P1). For, given (a,b), it is easy to construct a polynomial $Q_{a,b}$ on C such that

$$Q_{a,b}(\lambda+1)-Q_{a,b}(\lambda)=\Delta_b P(a+\lambda b) \qquad (\lambda\in C),$$
$$Q_{a,b}(0)=P(a).$$

Then $P(a+\lambda b)=Q_{a,b}(\lambda)$.

Example 1 shows that, in general, the degree $N_{a,b}$ of P_{ab} is not a bounded function on $G\times G$. Even on $C+C$, there exists a function satisfying (P1) with $N_{a,b}$ unbounded [4].

Proof of Theorem 1. *Sufficiency.* Let f be a function on G such that $\Delta_h f$ is a polynomial for each $h\in G$. Then f may be taken to be a continuous function, since otherwise there is an additive function Γ_1 on G such that $f-\Gamma_1$ is continuous, and the differences of $f-\Gamma_1$ will be polynomials. Also, f has property (P1) and therefore can (and will) be considered simply as a function on G/B.

Clearly, G/B contains a dense subgroup G' which has a finite Hamel basis, viz., $G'=R^m+G_1$, where R denotes the subgroup of E consisting of the rational numbers. If $H=\{h_1,\cdots,h_p\}$ is a Hamel basis for G', and if G'' is the group generated by H, then G'' is isomorphic to C^p. The isomorphism ϕ can be chosen so that $\phi h_i=\epsilon_i$ has δ_{ij} as its jth coordinate. Let polynomials f_i $(i=1,\cdots,p)$ be defined on C^p by

$$f_i(\phi x)=\Delta_{h_i}f(x) \qquad (x\in G'').$$

Then, clearly,

$$\Delta_{\epsilon_i}f_j=\Delta_{\epsilon_j}f_i \qquad (i,j=1,\cdots,p). \tag{2.2}$$

But (2.2) implies that there exists a polynomial g^* on C^p such that $g^*(x+\epsilon_i)-g^*(x)=f_i(x)$ $(i=1,\cdots,p)$. Explicitly, $g^*=g_p$, where g_k is given on C^k $(k=1,2,\cdots)$ by

$$\begin{aligned}&g_0\equiv 0,\\ &g_k(n_1,\cdots,n_{k-1},n_k)=g_{k-1}(n_1,\cdots,n_{k-1})\\ &\qquad+\sum_{m=0}^{M}c_{m,k}(n_1,\cdots,n_{k-1})(B_{m+1}(n_k)-B_{m+1})/(m+1).\end{aligned} \tag{2.3}$$

Here, the $c_{m,k}$ are the polynomials obtained from

$$f_k(n_1, \cdots, n_{k-1}, n_k, 0, \cdots, 0) = \sum_{m=0}^{M} c_{m,k}(n_1, \cdots, n_{k-1})(n_k)^m,$$

while $B_m(x)$ and B_m are the mth Bernoulli polynomial and number respectively. This assertion can be proved by induction on p. (Note that the well-known identity

$$(B_{m+1}(x+1) - B_{m+1}(x))/(m+1) = x^m$$

[10] implies that the sum (2.3) is

$$\left(-\sum_{j=n_k}^{-1} + \sum_{j=0}^{n_k-1} \right) f_k(n_1, \cdots, n_{k-1}, j), \tag{2.4}$$

the first (respectively, second) sum in (2.4) being empty if $n_k \geqq 0$ (resp., if $n_k \leqq 0$).)

Also, g^* has a unique extension as a polynomial g^{**} on all of R^p. Let g denote the function on G' given as follows: if

$$x \sim \sum_{i=1}^{p} r_i h_i, \quad \text{then } g(x) = g^{**}(r_1, \cdots, r_p).$$

Clearly, g satisfies (P1) and is of bounded degree. Since $\Delta_{h_i}(f-g)(x) = 0$ for each $x \in G''$ $(i = 1, \cdots, p)$, it follows that $f - g$ is constant on G''. If x is any point in G', then there exists a positive integer K such that $\mu K x \in G''$ for all μ in C. Then the polynomial in λ given by $f(\lambda x) - g(\lambda x)$, being constant for $\lambda = \mu K$ $(\mu \in C)$, is a constant on C, so that f is a polynomial on G'. If $N-1$ is its degree, then $\Delta_b^N f(a)$ vanishes for all $(a, b) \in G' \times G'$, and therefore, by continuity, for all $(a, b) \in G \times G$. But this implies that f is a polynomial of degree $N-1$ on G:

$$f(a+\lambda b) = \sum_{\mu=0}^{N-1} \sum_{\nu=0}^{\mu} (-1)^{\nu} \binom{\lambda}{\mu} \binom{\mu}{\nu} f(a+\nu b) \qquad (\lambda \in C; a, b \in G).$$

3. **The difference property for exponential polynomials.** A function z on G is a *generalized character* if z is a continuous homomorphism from G to the multiplicative group of nonzero complex numbers. A function e on G is an *exponential polynomial* if

$$e = \sum_{i=1}^{n} P_i z_i,$$

where each P_i is a polynomial and each z_i is a generalized character. If each P_i is a constant, and each z_i is an ordinary character, then e is said to be a *trigonometric polynomial.*

THEOREM 2. *Let G be an abelian locally compact group. A necessary and sufficient condition in order that the class of exponential polynomials on G have the difference property is that G be compactly generated.*

Let f be a function on G such that

$$\Delta_h f = \sum_{\alpha \in A} P_h^\alpha z_\alpha \qquad (h \in G) \tag{3.1}$$

where $\{z_\alpha : \alpha \in A\}$ is the set of all generalized characters on G, each P_h^α is a polynomial on G, and, for each fixed $h \in G$,

$$P_h^\alpha = 0 \qquad \text{for } \alpha \neq \alpha_1(h), \cdots, \alpha_{k(h)}(h). \tag{3.2}$$

Since the class of continuous functions on G has the difference property, f may be taken to be continuous.

Distinct generalized characters are linearly independent over the ring of polynomials on G (Lemma 3.1, below). Hence, f will be an exponential polynomial if and only if there exist polynomials $\{Q^\alpha : \alpha \in A\}$ such that

$$\Delta_h(Q^\alpha z_\alpha) = P_h^\alpha z_\alpha \qquad (h \in G, \alpha \in A), \tag{3.3}$$

and

$$Q^\alpha = 0 \qquad \text{for } \alpha \neq \alpha_1, \cdots, \alpha_k, \tag{3.4}$$

for then it is clear that

$$f = \sum_{\alpha \in A} Q^\alpha z_\alpha.$$

The proof of the sufficiency portion of Theorem 3 consists in constructing polynomials Q^α satisfying (3.3), and showing that (3.4) also holds. If G is not compactly generated, however, then it is not true that (3.2) and (3.3) imply (3.4), even when $\{\Delta_h f : h \in G\}$ are all trigonometric polynomials. This is shown in Theorem 3, from which the necessity of the condition in Theorem 2 will follow.

LEMMA 3.1. *Let G be an arbitrary group, let $z_1, \cdots, z_n$ be distinct homomorphisms of G into the multiplicative group of nonzero complex numbers, and let $P_1, \cdots, P_n$ be complex functions satisfying* (P1). *If*

$$P_1 z_1 + \cdots + P_n z_n \equiv 0, \tag{3.5}$$

then $P_1 \equiv \cdots \equiv P_n \equiv 0$.

Proof. First, consider the special case $G = C$. Since $z_1, \cdots, z_n$ are distinct, the complex numbers $z_1(1), \cdots, z_n(1)$ are necessarily distinct. If not all P_j are identically zero, then, reordering if necessary, it may be assumed that for some integer p, $1 \leqq p \leqq n$,

$$z_j(1)/z_1(1) = e^{i\beta_j} \neq 1 \qquad (\beta_j \text{ real}, 2 \leqq j \leqq p),$$
$$|z_1(1)| > |z_j(1)| \qquad (p+1 \leqq j \leqq n),$$
$$P_j(\lambda) = c_j\lambda^m + O(\lambda^{m-1}) \text{ as } \lambda \to \infty \qquad (1 \leqq j \leqq p; c_1 \neq 0).$$

It follows upon division of the terms of (3.5) by $\lambda^m z_1(\lambda)$ that

$$c_1 = -\sum_{j=2}^{p} c_j e^{i\beta_j\lambda} + O(\lambda^{-1}) \text{ as } \lambda \to \infty. \tag{3.6}$$

Thus, taking $\lambda = 1, 2, \cdots, N$ in (3.6), adding, and dividing by N, it is seen that

$$c_1 = -(1/N)\sum_{j=2}^{p} \frac{c_j e^{i\beta_j}(e^{i\beta_j N} - 1)}{e^{i\beta_j} - 1} + O(N^{-1}\log N). \tag{3.7}$$

Letting N approach infinity in (3.7), it follows that $c_1 = 0$, a contradiction. In the general case, we prove by induction on n that $P_1(x) \equiv 0$. This is obvious for $n = 1$, since z_1 is never zero. Now let $n > 1$, and suppose the result holds for all $p < n$. Choose $x_0 \in G$ such that $z_n(x_0) \neq z_1(x_0)$; this is possible since the z's are distinct. Suppose that $z_1(x_0) = z_2(x_0) = \cdots = z_p(x_0)$ for some integer p, $1 \leqq p < n$, while $z_j(x_0) \neq z_1(x_0)$ for $p < j \leqq n$. Then

$$\sum_{j=1}^{n} P_j(x + \lambda x_0) z_j(x + \lambda x_0) = \left[\sum_{j=1}^{p} P_j(x + \lambda x_0) z_j(x)\right] z_1(\lambda x_0)$$
$$+ \sum_{j=p+1}^{n} P_j(x + \lambda x_0) z_j(x + \lambda x_0).$$

For each fixed x, this expression can be considered as an exponential polynomial on C, and $z_1'(\lambda) = z_1(\lambda x_0)$ is distinct from the other generalized characters. Its coefficient is therefore zero for each $\lambda \in C$. Hence, for each $x \in G$

$$\sum_{j=1}^{p} P_j(x) z_j(x) = 0,$$

so that $P_1(x) \equiv 0$, by the induction assumption.

LEMMA 3.2. *Let G be an abelian topological group, let $z \not\equiv 1$ be a generalized character on G, and let $\{P_h: h \in G\}$ be a collection of polynomials on G such that*

$$P_h(x + h')z(h') - P_h(x) = P_{h'}(x + h)z(h) - P_{h'}(x), \tag{3.8}$$

for all h, h', x in G. Then there exists a polynomial Q on G such that

$$\Delta_h(Qz) = P_h z \qquad (h \in G). \tag{3.9}$$

Proof. Let $h \in G$ be such that $|z(h)| < 1$, or, if there is no h with this property, i.e., if $|z| \equiv 1$, let $z(h) \neq 1$. Consider the expression

$$\lim_{r\to 1-}\left\{-\sum_{n=0}^{\infty}(z(h)r)^nP_h(x+nh)\right\}\qquad (x\in G). \tag{3.10}$$

If $|z(h)|<1$, then the r in the summation may be replaced by 1, and the lim omitted. Since P_h is a polynomial,

$$P_h(x+nh)=\sum_{k=0}^{N}c_k(x)n^k\quad (x\in G), \tag{3.11}$$

where each $c_k(x)$ is a polynomial on G, obtainable explicitly from setting $n=0,1,\cdots,N$ in (3.11) and solving by Cramer's rule. The sum in (3.10) is given for $0<r<1$ by

$$-\sum_{k=0}^{N}c_k(x)\sum_{n=0}^{\infty}n^k(z(h)r)^n=-\sum_{k=0}^{N}c_k(x)\left\{y\frac{d}{dy}\right\}^k\left(\frac{1}{1-y}\right)\Bigg|_{y=rz(h)}.$$

Hence the limit in (3.10) exists and yields a polynomial on G; let this polynomial be denoted by Q. Let h' and x be arbitrary elements of G. Then

$$\begin{aligned}&Q(x+h')z(h')-Q(x)\\ &\quad=\lim_{r\to 1-}\left\{-\sum_{n=0}^{\infty}(z(h)r)^n[P_h(x+h'+nh)z(h')-P_h(x+nh)]\right\}.\end{aligned} \tag{3.12}$$

From (3.8), the right-hand side of (3.12) is

$$\begin{aligned}&\lim_{r\to 1-}\left\{-\sum_{n=0}^{\infty}(z(h)r)^n[P_{h'}(x+(n+1)h)z(h)-P_{h'}(x+nh)]\right\}\\ &\quad=P_{h'}(x)-\lim_{r\to 1-}(1-r)\sum_{n=1}^{\infty}P_{h'}(x+nh)z(h)^nr^{n-1}=P_{h'}(x),\end{aligned} \tag{3.13}$$

so that (3.9) follows from (3.12) and (3.13).

Proof of Theorem 2. *Sufficiency.* Let f be a function (which may and will be assumed to be continuous) such that (3.1) and (3.2) hold. From (3.1), Lemma 3.1, and the identity

$$\Delta_{h'}\Delta_hf=\Delta_h\Delta_{h'}f,$$

it follows that (3.8) holds for each generalized character z. For each z_α except $z_0\equiv 1$, it follows from Lemma 3.2 that there exists a polynomial Q^α satisfying (3.9); in particular, if $P_h^\alpha\equiv 0$ for all h, then $Q^\alpha\equiv 0$. But only finitely many Q^α can be nonzero. For suppose that $Q^i\not\equiv 0$ $(i=1,2,\cdots)$, with P_h^i and z_i the corresponding polynomials and generalized characters. For each $h\in G$, (3.2) shows that there exists an integer $i(h)$ such that $P_h^i\equiv 0$ for $i\geqq i(h)$. Hence, from (3.3),

$$Q^i(x+h)z_i(h)-Q^i(x)\equiv 0\qquad (i\geqq i(h),x\in G),$$

whence it follows that $z_i(h) = 1$ for $i \geqq i(h)$. Let

$$H_j = \{h: h \in G, z_i(h) = 1 \text{ for all } i \geqq j\}.$$

H_j is a closed subgroup of $G, H_{j+1} \supset H_j$ and $G = \bigcup H_j$, so that at least one H_j is of positive Haar measure, and thus open [5]. Therefore H_k is open for all $k \geqq j$. Let A be a compact neighborhood of 0 which generates G. Then $\bigcup \{H_k: k \geqq j\}$ covers G, so that A is covered by some H_N, whence $H_N = G$. Therefore $z_i \equiv 1$ for all $i \geqq N$, contradicting the distinctness of the z_i. Since $Q^\alpha \equiv 0$ for all but finitely many α, the function given by $g = \sum Q^\alpha z_\alpha$ (the summation taken for all α such that $z_\alpha \not\equiv 1$) is an exponential polynomial, and, for each $h, \Delta_h(f - g)$ is clearly a polynomial on G. But the function $f - g$ is continuous, and the class of polynomials on G has the difference property from Theorem 1, since G is compactly generated only if G_1 is finitely generated. Hence, $f - g$ is a polynomial on G.

In the proof just given, use was made of the fact that a compactly generated group G is not the countable union of a strictly increasing sequence of closed subgroups. Conversely,

Lemma 3.3. *If the locally compact abelian group G is not compactly generated, there is a sequence $\{H_j\}$ of closed subgroups of G, such that $H_j \subset H_{j+1}$ (strictly) and $\bigcup H_j = G$.*

Proof. There is a compact subgroup G' of G such that $G/G' = E^p + G_2$, with G_2 discrete. Since G is not compactly generated, it follows that G_2 is not finitely generated. It is known [9] that

$$G_2 = \bigcup_{n=1}^{\infty} S_n,$$

where each S_n is a direct sum of cyclic groups, and $S_n \subset S_{n+1}$. If the inclusion is proper for infinitely many n, the choice of the H_j is clear, and the lemma follows. Otherwise, G_2 is itself a direct sum of infinitely many cyclic groups:

$$G_2 = \sum_\alpha A_\alpha.$$

Let $\{A_{\alpha_1}, A_{\alpha_2}, \cdots\}$ be a countably infinite subset of $\{A_\alpha\}$, and let

$$H_j = E^p + \sum \{A_\alpha: \alpha \neq \alpha_{j+1}, \alpha_{j+2}, \cdots\}.$$

Then $H_j \subset H_{j+1}$ properly, and their union is G.

Theorem 3. *Let G be an abelian locally compact group. The class of trigonometric polynomials on G has the difference property if and only if G is compactly generated.*

Proof. The sufficiency is clearly a corollary of the sufficiency proof of Theorem 2. If G is not compactly generated, let $\{H_j\}$ be the sequence given by Lemma 3.3, and for each j let z_j be a character identically 1 on H_j but not identically 1 on G; such characters exist [11]. Let $\sum a_j$ be a convergent infinite series of positive numbers, and let f be defined by

$$f = \sum_{j=1}^{\infty} a_j z_j. \tag{3.14}$$

If $h \in G$ is given, there exists an integer $k = k(h)$ such that $h \in H_{k+1}$. Then

$$f(x + h) - f(x) = \sum_{j=1}^{k} a_j(z_j(h) - 1)z_j(x),$$

a trigonometric polynomial.

Suppose now that f, given by (3.14) is also given by

$$f = \sum_{i=1}^{n} P_i z_{\alpha_i} + \Gamma, \tag{3.15}$$

with polynomials P_i, generalized characters z_{α_i}, and an additive function Γ. Let z_j be a character appearing in (3.14) but not in (3.15), and let $h \in G$ be chosen such that $z_j(h) \neq 1$. Then z_j appears in the expression for the exponential polynomial $\Delta_h f$ obtained from (3.14) but not in that obtained from (3.15). This contradicts Lemma 3.1. Thus the necessity portions of both Theorem 2 and of Theorem 3 are established.

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